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B.Sc. Sem-4. MJC

19/2/26

Que 1:- State and Prove Darboux's theorem.

Let f be a bounded function on a closed bounded interval $[a,b]$. Then to every $\epsilon > 0$ there corresponds $\delta > 0$ such that

$$U(P) < \overline{\int_a^b} f(x)\,dx + \epsilon \;\&\; L(P) > \underline{\int_a^b} f(x)\,dx - \epsilon$$

for all partitions P of $[a,b]$ with norm $\mu(P) < \delta$

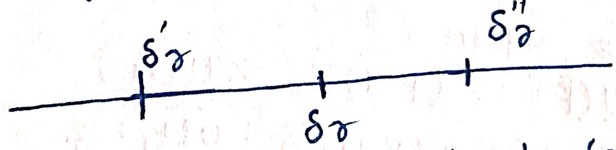
Lemma: Let $|f(x)| \le K$ for all $x \in [a,b]$

Let δ be a +ve number and let P_1 be a partition of $[a,b]$ such that $\mu(P_1) \le \delta$.

Let P_2 be a partition of $[a,b]$ consisting of all the points of P_1 and at the most some p more. Then we have

$$U(P_1) - 2pK\delta \le U(P_2) \le U(P_1)$$

Proof of the Lemma:- First suppose that $p=1$ so that only one interval say δ_r of P_1 is divided into two subintervals δ'_r and δ''_r. Let M_r, M'_r, M''_r be the



We have,
$$U(P_1) - U(P_2) = M_r \cdot |\delta_r| - (M'_r|\delta'_r| + M''_r|\delta''_r|)$$
$$= (M_r - M'_r)|\delta'_r| + (M_r - M''_r)|\delta''_r|$$

for $|\delta_r| = |\delta'_r| + |\delta''_r|$

Now since $|f(x)| \le K$ for all $x \in [a,b]$

therefore
$$-K \le M'_r \le M_r \le K \quad \text{i.e.} \quad 0 \le M_r - M'_r \le 2K$$

Similarly
$$0 \le U(P_1) - U(P_2) \le 2K(|\delta'_r| + |\delta''_r|)$$
$$= 2K|\delta_r| \le 2K\delta$$

Now supposing that each additional point is introduced one by one, we obtain the result

We now prove the main theorem:

As F is bounded $\exists K > 0$ such that

$$|F(x)| \leq K \text{ for all } x \in [a,b]$$

Since $\overline{\int_a^b} F(x).dx = glb\{U(P)\}$

$\exists$ a Partition $P_1 = \{a = x_0, x_1, \dots, x_p = b\}$ such that

$$U(P_1) < \overline{\int_a^b} F(x).dx + \frac{\epsilon}{2}$$

The Points of P_1 are $(p+1)$ in number. Let δ be the +ve number

Such that

$$2K(p-1).\delta = \frac{\epsilon}{2}.$$

Let P be any Partition of $[a,b]$ with $\mu(P) \leq \delta$

Let P_2 be the Common refinement of P_1 and P

From the above lemma, we have

$$U(P) - 2(p-1)K\delta \leq U(P_2) \leq U(P)$$

Also $\quad U(P_2) \leq U(P_1)$

Thus we obtain

$$U(P) - 2(p-1)K\delta \leq U(P_1)$$

$$\therefore \quad U(P) \leq 2(p-1)K\delta + U(P_1)$$

$$< \frac{\epsilon}{2} + \overline{\int_a^b} F(x).dx + \frac{\epsilon}{2} = \overline{\int_a^b} F(x).dx + \epsilon$$

Similary it Can be proved that

$$L(P) > \underline{\int_a^b} F(x).dx - \epsilon$$

for all Partitions P with

$$\mu(P) \leq \delta.$$